Spatially resonant interactions in annular convection

Arantxa Alonso, Marta Net and Juan Sánchez

Abstract. Different types of steady columnar patterns in an annular container with a fixed value of the radius ratio are analyzed for a low Prandtl number Boussinesq fluid. The stability of these convection patterns as well as the spatial interaction between them resulting in the formation of mixed modes are numerically investigated by considering the original nonlinear set of Navier-Stokes equations. A detailed picture of the nonlinear dynamics before temporal chaotic patterns set in is presented and understood in terms of symmetry-breaking bifurcations in an $\mathbf{O}(2)$-symmetric system. Special attention is paid to the strong spatial 1:2 resonance of the initially unstable modes with wavenumbers n=2 and n=4, which leads to bistability in the system.

1. Introduction

In the present paper we investigate the process of pattern selection and mode interaction in the context of two-dimensional thermal convection. We analyze convection in a rotating annulus with gravity radially inwards and outwards heating, restricting our attention to exactly two-dimensional solutions. These solutions, which form columns parallel to the axis of rotation, are allowed when stress-free boundary conditions on the lids of the annulus are considered, and are the preferred modes at the onset of convection for large enough rotation rates [1].

When the two-dimensional governing equations are considered the symmetry of the system is $\mathbf{O}(2)$ even in the rotating case. Although rotation breaks the reflection symmetry in vertical planes containing the axis, for the columnar solution the Coriolis term can be written as a gradient and introduced in the pressure term. Rotation drops from the equations and they retrieve the reflection invariance. Thus, columnar convection in a rotating annulus provides a simple fluid dynamics system with $\mathbf{O}(2)$ symmetry, which exhibits a rich variety of stationary and spatio-temporal patterns [2], [3], [4].

In a previous work [2], we studied the transition route to chaos that the stable columnar solution with wavenumber n=3 undergoes for a low value of the Prandtl number, σ=0.025, and a fixed value of the radius ratio, η=0.3. The steady columns give rise to spatially periodic and non-periodic direction reversing travelling waves, which become chaotic through a Neimark-Sacker subcritical bifurcation. In order to complete the analysis, we will now focus on the unstable branches that bifurcate from the conduction state. We will see that there is a strong spatial interaction between the two initially unstable modes with wavenumbers n=2 and n=4 and that some aspects of the behaviour we find, such as the presence of a mixed mode, wavenumber gaps and travelling waves, are predicted by the normal form equations for the 1:2 spatial resonance with $\mathbf{O}(2)$ symmetry [5], [6].

2. Formulation of the problem

We consider the problem of nonlinear convection in a cylindrical annulus with radius ratio $\eta = r_i/r_o$, where r_i and r_o are the inner and outer radii, rotating about its axis of symmetry, filled with a Boussinesq fluid of thermal diffusivity κ, thermal expansion coefficient α and kinematic viscosity ν. The inner and outer rigid sidewalls are maintained at constant temperatures T_i and T_o, with $T_i > T_o$, and the gravitational acceleration is taken radially inwards, $\mathbf{g} = -g\hat{\mathbf{e}}_r$, and is assumed to be constant.

There exists a basic conduction state in which heat is radially transferred towards the outer cylinder by thermal conduction and which shares the full symmetries of the system. The stability of this state is determined by the Navier-Stokes, continuity and heat equations. When horizontal stress-free lids are considered, the linear stability analysis shows that there is always a moderate rotation rate above which steady exactly two-dimensional columns parallel to the axis of the annulus are the preferred solutions at the onset of convection [1]. These solutions are characterized by a fixed azimuthal wavenumber, n, which is imposed by the chosen radius ratio.

In order to obtain the nonlinear steady columnar solutions and to analyze their stability with respect to axial independent disturbances as any parameter of interest is varied, we have developed a continuation code [7]. To solve the equations we have used a technique based on velocity potentials. In this formulation the velocity field is written as $\mathbf{u} = f\hat{\mathbf{e}}_\theta + \nabla \times \Psi\hat{\mathbf{e}}_z$, where Ψ=$\Psi(r,\theta)$ is a function which has zero azimuthal average. We are considering an explicit equation for $f = f(r)$, which is the simplest way of including the possibility of generating a mean mass flow in the azimuthal direction (average of the azimuthal velocity in the radial direction). The resulting nonlinear equations in nondimensional form are

$$\partial_t f = \sigma\nabla^2_- f + P_\theta\Big[\nabla^2_h\Psi\big(\frac{1}{r}\partial_\theta\Psi\big)\Big], \tag{1a}$$

$$\begin{aligned}\partial_t\nabla^2_h\Psi &= \sigma\nabla^4_h\Psi + (1-P_\theta)\frac{\sigma Ra}{r}\partial_\theta\Theta + (1-P_\theta)J(\Psi,\nabla^2_h\Psi) + \\ &\quad +\nabla^2_- f\big(\frac{1}{r}\partial_\theta\Psi\big) - f\big(\frac{1}{r}\partial_\theta\nabla^2_h\Psi\big),\end{aligned} \tag{1b}$$

$$\partial_t\Theta = \nabla^2_h\Theta - \frac{1}{r^2\ln\eta}\partial_\theta\Psi + J(\Psi,\Theta) - f\big(\frac{1}{r}\partial_\theta\Theta\big), \tag{1c}$$

where Θ denotes the departure of the temperature from its conduction profile and $\nabla_{-}^{2} = \partial_r(\partial_r + 1/r)$. P_θ is the projection operator that extracts the zero-azimuthal mode in a Fourier expansion, J is the jacobian in cylindrical coordinates and Ra and σ are the Rayleigh and Prandtl numbers. The variables have been expanded in terms of Chebyshev polynomials and Fourier expansions and no-slip and perfectly conducting boundary conditions on the lateral walls have been considered.

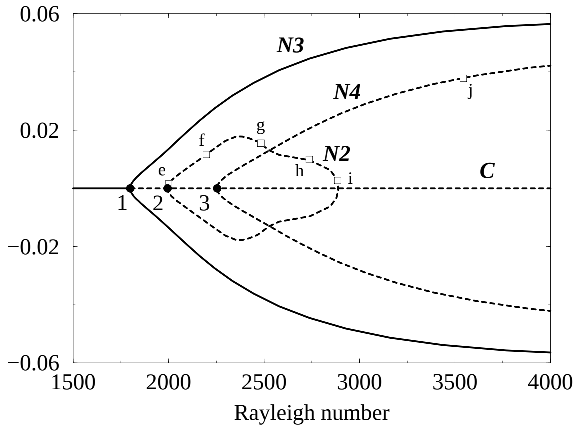
3. Nonlinear steady columnar solutions: results and discussion

In this section we describe the results for η=0.3, σ=0.025 and increasing Rayleigh number. We obtain the steady columnar patterns that bifurcate from the conduction state and analyze their stability.

The results are summarized in figure 1. In the upper part of the figure, the bifurcation diagram shows the branches of columnar solutions with basic wavenumbers n=3,2,4 ($N3$, $N2$ and $N4$ branches, respectively). In the diagram, we are plotting an amplitude of the dominant mode in each case. The conduction state becomes unstable to columns with wavenumber n=3 at Ra_c^1=1799 (point 1 in the bifurcation diagram), in agreement with the linear stability analysis. For slightly larger Rayleigh numbers, the conduction state is also unstable to modes with wavenumber n=2 (at Ra_c^2=1995) and n=4 (at Ra_c^3=2254). The new nonaxisymmetric solutions break the rotation symmetry, $\mathbf{R}_\theta$, of the basic state, but maintain the reflection symmetry, $\mathbf{R}_1$, with respect to appropiate vertical planes $\theta = \theta_0$ and the invariance under $2\pi/n$-rotations, $\mathbf{R}_{2\pi/n}$. The group of symmetry of the new solutions is $\mathbf{D}_n$. Thus, bifurcations from the conduction state are symmetry-breaking steady-state bifurcations in which multiplicity two eigenvalues cross the imaginary axis.

Whereas solutions along the $N3$ and $N4$ branches are pure modes, in which only the basic wavenumbers and their harmonics are nonzero, the $N2$ is a mixed-mode branch. There is a strong spatial interaction between the n=2 and n=4 modes which produces a change in the structure of the solution along the $N2$ branch. To illustrate the physical nature of these solutions, the lower part of figure 1 shows the temperature contour plots at different Rayleigh numbers. As the Rayleigh number increases, the contribution of the n=4 mode becomes more and more important, while the n=2 contribution diminishes until vanishing. The initial two pairs of rolls become a n=4 solution.

A stability analysis of the mixed-mode solutions shows that there are several bifurcations in the $N2$ branch. The new branches have been included in figure 2. Bifurcation points 4 (Ra_c^4=2362), 6 (Ra_c^6=2509) and 7 (Ra_c^7=2712) correspond to subharmonic steady-state bifurcations. The solutions in these new branches, which are displayed in the right-hand side of figure 2, still keep the reflection symmetry between columns, but now there is a contribution of all the wavenumbers. Their



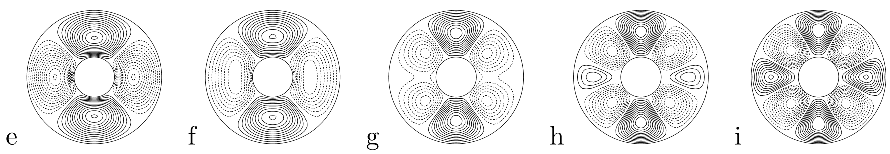


FIGURE 1. (top) Bifurcation diagram showing branches of columnar modes with wavenumbers n=3, n=2 and n=4 ($N3$, $N2$ and $N4$ branches, respectively). They are born at Ra_c^1=1799 (1), Ra_c^2=1995 (2) and Ra_c^3=2254 (3). (bottom) Temperature contour plots showing the evolution of the columns on the $N2$ branch with increasing Rayleigh number. They correspond to points e (Ra=2000), f (Ra=2198), g (Ra=2500), h (Ra=2711) and i (Ra=2875) in the diagram.

group of symmetry is $\mathbf{Z}_2$. The bifurcation identified in point 5 (Ra_c^5=2478) corresponds to a steady-state instability that keeps the wavenumber of the main solution, n=2, but in which the mean flow becomes nonzero. According to bifurcation theory [8], a steady-state bifurcation that breaks the reflection symmetry keeping the rotational invariance would give rise to travelling waves with zero phase speed at the bifurcation point. Nevertheless, we have not followed this time-dependent branch which, in our case, is unstable. Finally, two subsequent bifurcations very close to each other take place in the neighbourhood of point 8. In the first one (Ra_c^8=2887.5), one of the two positive eigenvalues of the solution is stabilized through a subharmonic steady-state bifurcation. In the second one ($Ra_c^{8'}$=2888.9) the amplitude of the n=2 mode vanishes. The $N2$ branch joins the $N4$ branch, and columns with wavenumber n=2 cease to exist. This is a bifurcation from the $N4$ branch, which takes place after a bifurcation in Ra_c^9=2851 in which an eigenvalue with multiplicity two gains stability.

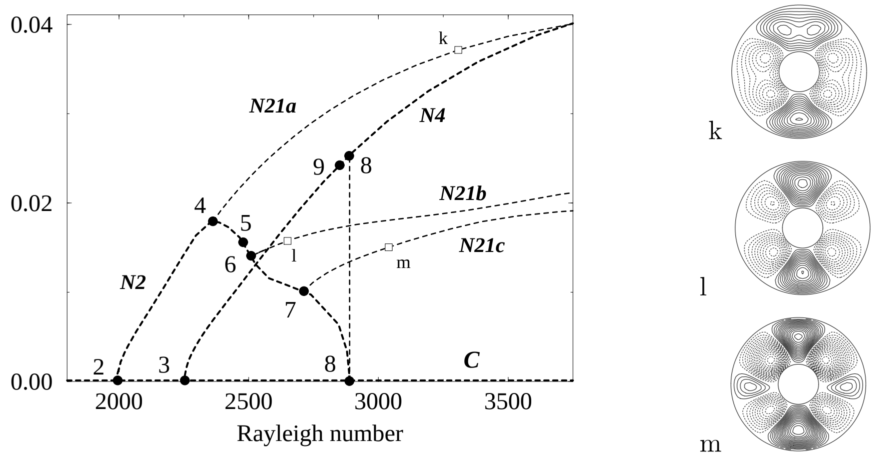


FIGURE 2. (left) Detail of the steady-state bifurcations on the $N2$ and $N4$ branches, which take place at Ra_c^4=2362 (4), Ra_c^5=2478 (5), Ra_c^6=2509 (6), Ra_c^7=2712 (7) and Ra_c^8=2888 (8) on the $N2$ branch and at Ra_c^9=2851 (9) and Ra_c^8=2888 (8) on the $N4$ branch. (right) Temperature contour plots showing the structure of the solutions in the $N21a$, $N21b$ and $N21c$ branches corresponding to points k (Ra=3308), l (Ra=2649) and m (Ra=3040) in the bifurcation diagram.

All the steady patterns above described except for the n=3 column are unstable. However, by extending further the $N4$ branch, a bifurcation that stabilizes the n=4 mode by shedding a new unstable n=2 branch takes place at Ra_c^{10}=4779. As a result, for Rayleigh numbers larger than Ra_c^{10} at least two stable solutions coexist: steady columns with wavenumber n=4 and direction reversing travelling waves with wavenumber n=3.

The spatial interaction between the modes with wavenumbers n=2 and n=4 that we have found is an example of an 1:2 resonance, in which modes with wavenumbers n and $2n$ in the periodic direction interact nonlinearly. The 1:2 resonance with $\mathbf{O}(2)$ symmetry was first studied by Dangelmayr [5] and some aspects of the dynamics predicted by the normal form equations are reproduced here. First, the presence of wavenumber gaps in which no steady solutions with a given wavenumber exist is a typical feature of this resonance. This is what happens in the range of Rayleigh numbers $2888 < Ra < 4779$, where the n=2 solution disappears. The existence of travelling waves bifurcating from the n-mode, which correspond to the bifurcation point at Ra_c^5=2478 in our case, is also an identity sign of this resonance.

Dynamics dominated by the strong spatial 1:2 resonance are expected in systems without midplane layer symmetry. In the case of annular convection this symmetry is broken by curvature, while in two-dimensional Rayleigh-Bénard convection, the same effect can be achieved by considering different boundary conditions at top and bottom or by including non-Boussinesq terms [9]. For instance, the analysis of a long-wave model for two-dimensional convection in a plane layer shows that the strong 1:2 resonance is dominant when asymmetric boundary conditions are considered and a behaviour similar to the one we find in annular convection is described [10]. In contrast, in Rayleigh-Bénard convection with symmetric boundary conditions the leading order resonant term in the 1:2 interaction is of higher order than that in the 1:3 interaction [10], [11], [12].

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Departament de Física Aplicada,
Universitat Politècnica de Catalunya,
Campus Nord, Mòdul B4
08034 Barcelona, Spain
E-mail address: arantxa@fa.upc.es