

Research paper

Nested Feigenbaum cascades and Shil'nikov-like dynamics in convection of binary mixtures

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ABSTRACT

Numerical continuation and stability analysis of periodic orbits are used to analyze, under a dynamical systems point of view, the time dependent dynamics that arise from the steady convection of a H₂-Xe gas mixture, driven by lateral temperature and low concentration gradients. Both Soret and Dufour effects are taken into account. A bifurcation diagram consisting of sequences of period-doubling and saddle-node bifurcations is unfolded. They give rise to a series of nested Feigenbaum cascades in a narrow region of the parameter space. Several types of cyclic and chaotic solutions close to heteroclinic orbits were found in the interval of the control parameter where the periodic orbits are unstable. The relations between them are established.

1. Introduction

Double-diffusive convection is present in a large number of natural phenomena in different fields, such as oceanography, astrophysics, geophysics, etc., and in many industrial processes, such as crystal growth and solidification of binary mixtures in general, prevention of subsoil water pollution, metallurgy, storage of nuclear wastes, etc., where to avoid or to promote periodic and chaotic oscillations through especial designs, vibration, or flow rate controls can be essential.

A large amount of studies on the subject have been devoted to the steady convection, mainly those related to channels and slots driven by lateral gradients of heating and concentration [1–8]. Much less attention has been paid to the study of flows with periodic or complex time dependence [9–11], and even less if the Soret and Dufour effects are taken into account.

The onset and behavior of periodic oscillations was recently studied in a slot of aspect ratio $\Gamma = 8$ for several set of parameters [11]. It was shown that, for a H₂-Xe gas mixture of separation ratio $Se = 0.232$, Dufour coefficient $Q = 0.357$ [12], Rayleigh number ratio $N = Ra^T/Ra^C$ (Ra^T being the thermal and Ra^C the solutal Rayleigh numbers, respectively), and $Ra^C = 1$, the onset of the oscillations takes place at a Hopf bifurcation on the main branch of steady flows, keeping their center-symmetry. The equilibria near the bifurcation point consist of a large convective cell filling the box with two strong vortices inside, one in each half of the cavity (see Fig. 2 in Ref. [11]). The Hopf bifurcation leads to a Feigenbaum cascade (FC) of periodic orbits (POs), but the complex time dependent dynamics was not explored in detail. The POs of any period consist mainly in the vertical displacement of the vortices together with the formation of a weak central vortex when the other two are near the ends of the cavity.

This paper is devoted to report the relation between a sequence of FCs, found by means of a Newton-Krylov continuation method of POs, and the Shil'nikov-like dynamics existent in the interval of the control parameter $N = Ra^T$ where the POs are unstable. Direct numerical simulation of the flow is used for the latter purpose.

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After the introduction a summary of the equations and numerical methods detailed in Ref. [11] is included in [Section 2](#). [Section 3](#) describes the bifurcation diagram of the nested FCs, and [Section 4](#) analyzes the stable complex dynamics present in the range of parameters where the FCs are confined, and [Section 5](#) summarizes and discusses the results.

2. Mathematical formulation

The dynamics of time periodic and time complex double-diffusive convection is analyzed numerically in a slot Ω of height h , width d , and aspect ratio $\Gamma = h/d$. It is heated or cooled from the lateral sides, which are assumed to maintain a constant temperature and concentration. The top and bottom boundaries are taken impermeable and insulating. The gravity is vertical, pointing downwards. The Navier-Stokes, mass, energy and concentration equations including the Soret and Dufour effects, were derived in [11,13], following [14,15], with the heat and mass fluxes written as in [16] but neglecting the barodiffusion term, i.e. as

$$\mathbf{J}_c = -\rho D \left(\nabla C^* + \frac{K_T}{T^*} \nabla T^* \right), \quad (1)$$

$$\mathbf{J}_q = \left[K_T \left(\frac{\partial \mu}{\partial C^*} \right)_{p^*, T^*} - T^* \left(\frac{\partial \mu}{\partial T^*} \right)_{p^*, C^*} + \mu \right] \mathbf{J}_c - K \nabla T^*, \quad (2)$$

where T^* and C^* mean the dimensional temperature and concentration before the scaling, and μ the chemical potential, D is the mass diffusion coefficient, K_T the thermal diffusion ratio, K the thermal conductivity, ρ the density, and p^* the pressure, which is taken constant.

Small variations of T^* and C^* are considered to find the Boussinesq approximation of the equations, then $\mathbf{J}_c$ and $\mathbf{J}_q$ are taken proportional to ∇C^* and ∇T^* in

$$\rho \frac{dC^*}{dt} = -\nabla \cdot \mathbf{J}_c, \quad (3)$$

$$\rho T^* \frac{dS^*}{dt} = \phi - \nabla \cdot (\mathbf{J}_q - \mu \mathbf{J}_c) - \mathbf{J}_c \cdot \nabla \mu, \quad (4)$$

where ϕ is the viscous dissipation function, and S^* the entropy of the system. By substituting the fluxes into [Eqs. \(3\)](#) and [\(4\)](#), transforming $\partial S^*/\partial t$ as a function of T^* , canceling high order terms, taking as scales d for the length, the lateral difference of temperature and concentration, ΔT^* , and ΔC^* , for the temperature and concentration, d^2/κ for the time, κ being the thermal diffusivity, and finally splitting the nondimensional $T = (1-x) - \Theta$ and $C = (1-x) - \Sigma$, the nondimensional equations

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \Pi + \text{Pr} \Delta \mathbf{v} - \text{Ra}^T \text{Pr} (x - \Theta) \mathbf{j} + \text{Ra}^C \text{Le} (x - \Sigma) \mathbf{j}, \quad (5)$$

$$\partial_t \Theta + (\mathbf{v} \cdot \nabla) \Theta = (1 + \text{Le} Q \text{Se}^2) \Delta \Theta + \frac{\text{Le}^2 Q \text{Se}}{\text{NPr}} \Delta \Sigma + \nu_x, \quad (6)$$

$$\partial_t \Sigma + (\mathbf{v} \cdot \nabla) \Sigma = \text{Le} \Delta \Sigma + \text{Pr} \text{Se} \text{N} \Delta \Theta + \nu_x, \quad (7)$$

are obtained.

The streamfunction formulation with the velocity field related to the streamfunction, ψ , by $\mathbf{v} = (\partial_y \psi, -\partial_x \psi)$, and non-slip boundary conditions everywhere are used here. [Eqs. \(5\)–\(7\)](#) give

$$\partial_t \Delta \psi + J(\Delta \psi, \psi) = \text{Pr} \Delta^2 \psi + \text{Ra}^T \text{Pr} (1 - \partial_x \Theta) - \text{Ra}^C \text{Le} (1 - \partial_x \Sigma), \quad (8)$$

$$\partial_t \Theta + J(\Theta, \psi) = (1 + \text{Le} Q \text{Se}^2) \Delta \Theta + \frac{\text{Le}^2 Q \text{Se}}{\text{NPr}} \Delta \Sigma + \partial_y \psi, \quad (9)$$

$$\partial_t \Sigma + J(\Sigma, \psi) = \text{Le} \Delta \Sigma + \text{Pr} \text{Se} \text{N} \Delta \Theta + \partial_y \psi, \quad (10)$$

with $J(f, h) = \partial_x f \partial_y h - \partial_y f \partial_x h$.

The system depends on six nondimensional physical parameters, the thermal and solutal Rayleigh numbers, Ra^T and Ra^C , the Prandtl, Lewis and Dufour numbers, Pr , Le and Q , and the separation ratio, Se .

$$\text{Ra}^T = \frac{\alpha g \Delta T^* d^3}{\nu \kappa}, \quad \text{Ra}^C = \frac{\beta g \Delta C^* d^3}{D \kappa}, \quad \text{Pr} = \frac{\nu}{\kappa}, \quad \text{Le} = \frac{D}{\kappa},$$

$$Q = \frac{\overline{T^*} \alpha^2}{c_p \beta^2} \left(\frac{\partial \mu}{\partial C^*} \right)_{p^*, T^*}, \quad \text{Se} = \frac{\beta K_T}{\alpha T^*}.$$

The set of boundary conditions is:

$$\psi = \partial_n \psi = 0 \quad \text{on} \quad \partial \Omega, \quad (11)$$

$$\Theta = 0 \quad \text{on} \quad x = 0, 1 \quad \text{and} \quad \partial_y \Theta = 0 \quad \text{on} \quad y = 0, \Gamma, \quad (12)$$

$$\Sigma = 0 \quad \text{on} \quad x = 0, 1 \quad \text{and} \quad \partial_y \Sigma = 0 \quad \text{on} \quad y = 0, \Gamma, \quad (13)$$

where $(x, y) \in \Omega = [0, 1] \times [0, \Gamma]$ because d is taken as longitude scale, and n indicates the normal direction to the boundary. This set of equations and boundary conditions does not have a solution with $\mathbf{v} = 0$ for arbitrary values of the parameters.

The system is $\mathcal{Z}_2$ equivariant, i.e., the equations and boundary conditions remain invariant under the center symmetry

$$S : (t, x, y, \psi, \Theta, \Sigma) \rightarrow (t, 1-x, \Gamma-y, \psi, -\Theta, -\Sigma). \quad (14)$$

The steady and periodic orbits have been calculated with a Newton-Krylov method. Assume that

$$\dot{z} = f(z, p), \quad (z, p) \in U \subset \mathbb{R}^n \times \mathbb{R} \quad (15)$$

is a generic large-scale autonomous system of ordinary differential equations depending on a parameter p , obtained after the spatial discretization of a system of partial differential equations [Eqs. (5)–(7) in our case] and that $\varphi(t, z, p)$ is its solution with initial condition z at $t = 0$ for a fixed value of p , i.e. $\varphi(0, z, p) = z$. A trajectory $\varphi(t, z_0, p)$ starting at z_0 , is a periodic orbit if there is a minimal $T > 0$ (its period) such that $z_0 - \varphi(T, z_0, p) = 0$. Since this equation is satisfied by any point on the periodic orbit, an additional scalar condition $g(z, p) = 0$ (phase condition) must be added to determine a single one and the period T . The final system of equations to solve is

$$z - \varphi(T, z, p) = 0, \quad (16)$$

$$g(z, p) = 0. \quad (17)$$

In most cases $g(z, p) = 0$ is the equation of a hyperplane cutting transversely the periodic orbits. It can be fixed or varied with the parameter p . The equilibria can also be found by solving the system $z - \varphi(T, z, p) = 0$ where T is now a fixed integration time, and no additional condition is required.

The stability of both equilibria and periodic orbits can be found by integrating during the time T the linearized (first variational) equation coupled with the original system

$$\dot{z} = f(z, p), \quad z(0) = z_0, \quad (18)$$

$$\dot{w} = D_z f(z, p)w, \quad w(0) = \delta z. \quad (19)$$

where $z(0) = z_0$ is a point of the periodic orbit or an equilibrium (in the latter case the original system needs not be integrated), and $w(0) = \delta z$ is a generic perturbation of z_0 . If $\varphi_1(t, \delta z, p)$ is the solution of Eq. (19) with initial condition δz the eigenvalues of the linear map $\delta z \rightarrow \varphi_1(T, \delta z, p)$ are the Floquet multipliers of the periodic orbit if z_0 is one of its points. If it is an equilibrium, the logarithms of the eigenvalues divided by T are the eigenvalues of the Jacobian $D_z f(z_0, p)$ at z_0 .

The nonlinear systems are solved by Newton's method, and at each step the linear systems are solved by GMRES(M) [17]. This defines a Newton-Krylov method. For the stability the eigenvalues of largest modulus are obtained by an Arnoldi method, which is an improvement of the power method, using the ARPACK package [18]. More detailed information on these techniques can be found in Refs. [9,19–21].

To approximate the spatial differential operators collocation methods are used in both x and y directions on a product mesh of Gauss-Lobatto points. The stiff system of ODEs obtained after the spatial discretization is integrated by means of an implicit-explicit (IMEX) third-order BDF-extrapolation method.

The first variational equations for the perturbations $(\psi_1, \Theta_1, \Sigma_1)$, and δRa^T , needed to apply the Newton-Krylov continuation and stability analysis methods when Ra^T is the control parameter are

$$\begin{aligned} \partial_t \Delta \psi_1 + J(\Delta \psi_1, \psi) + J(\Delta \psi, \psi_1) &= \text{Pr} \Delta^2 \psi_1 - \text{Ra}^T \text{Pr} \partial_x \Theta_1 + \text{Ra}^C \text{Le} \partial_x \Sigma_1 + \\ &\quad \delta \text{Ra}^T \text{Pr} (1 - \partial_x \Theta) \end{aligned} \quad (20)$$

$$\begin{aligned} \partial_t \Theta_1 + J(\Theta_1, \psi) + J(\Theta, \psi_1) &= (1 + \text{Le} Q \text{Se}^2) \Delta \Theta_1 + \frac{\text{Le}^2 Q \text{Se}}{N \text{Pr}} \Delta \Sigma_1 - \\ &\quad \delta \text{Ra}^T \frac{\text{Le}^2 Q \text{Se}}{\text{Ra}^T N \text{Pr}} \Delta \Sigma + \partial_y \psi_1, \end{aligned} \quad (21)$$

$$\partial_t \Sigma_1 + J(\Sigma_1, \psi) + J(\Sigma, \psi_1) = \text{Le} \Delta \Sigma_1 + N \text{Pr} \text{Se} \Delta \Theta_1 + \delta \text{Ra}^T \frac{\text{Pr} \text{Se}}{\text{Ra}^C} \Delta \Theta + \partial_y \psi_1, \quad (22)$$

with boundary conditions

$$\psi_1 = \partial_n \psi_1 = 0 \quad \text{on} \quad \partial \Omega. \quad (23)$$

$$\Theta_1 = 0 \quad \text{on} \quad x = 0, 1 \quad \text{and} \quad \partial_y \Theta_1 = 0 \quad \text{on} \quad y = 0, \Gamma, \quad (24)$$

$$\Sigma_1 = 0 \quad \text{on} \quad x = 0, 1 \quad \text{and} \quad \partial_y \Sigma_1 = 0 \quad \text{on} \quad y = 0, \Gamma. \quad (25)$$

The period of the POs, T , and the Nusselt number, $\overline{\partial_x T}$, calculated at $x = 1$ are used to illustrate the results. The latter is defined as

$$\overline{\partial_x T} = \frac{1}{T} \int_0^{t+T} \frac{1}{\Gamma} \int_0^\Gamma \partial_x T \, dy \, dt. \quad (26)$$

For the POs T is the period, and for the equilibria the temporal average is missing.

Some comparisons with previous results already published on double-diffusive convection, closely related to our computations, were done to check the numerical codes. The agreement with the critical parameter $\text{Ra}^T |\text{Le}^{-1} - 1|$ and eigenfunctions given in Refs. [1,2] in the balanced buoyancy case ($\text{Ra}^T \text{Pr} = \text{Ra}^C \text{Le}$) is excellent. We have checked all the aspect ratios tabulated. For instance, $\text{Ra}^T |\text{Le}^{-1} - 1| = 6991.37$ for $\Gamma = 4$, and our codes give 6991.3676. For an infinite vertical slot with periodic boundary conditions their $\text{Ra}^T |\text{Le}^{-1} - 1| = 6509 \pm 2$. Our extrapolation to $\Gamma \rightarrow \infty$ using a potential fitting with just seven points from $\Gamma = 7$ to 18, but with top and bottom nonslip boundary conditions, gives the function $\text{Ra}^T |\text{Le}^{-1} - 1| = A + B\Gamma^p$, with $A = 6511$, $B = 8751$ and $p = -2.05$ and errors $\varepsilon^A = 0.03\%$, $\varepsilon^B = 3.4\%$, and $\varepsilon^p = 12.5\%$, agreeing with their critical value. The nonlinear least-squares Marquardt-Levenberg algorithm was used to fit the function.

Table 1

Critical Ra_i^T , $i = 0, 1, 2, \dots, 7$ of the beginning of the branches, of the bifurcation points on the PD, PD2a, PD3a, ... branches of Fig. 1 (starting and following the branches from left to right), and of the final point. The dashed-line entries indicate that the corresponding critical Ra^T have not been computed with precision. The parameters $Ra^C = 1$, $Pr = 0.168$, $Le = 1.238$, $Se = 0.232$, $Q = 0.357$, and $\Gamma = 8$ are kept constant.

Branch	Ra_0^T	Ra_1^T	Ra_2^T	Ra_3^T	Ra_4^T	Ra_5^T	Ra_6^T	Ra_7^T
PD	4030.7	4155.1	4533.1	4539.0	4327.5	4328.8		
PD2a	4155.1	4183.1	4425.6	4425.7	4294.1	4294.5	4529.9	4533.1
PD3a	4183.1	4189.2	4377.532	4377.533	4287.979	4287.984	4425.47	4425.6
PD3c	4294.5	4294.8	4391.889	4391.890	4299.854	4299.856	4529.2	4529.9
PD4a	4189.2	4190.5	---	4382.67	4288.51	---	---	4425.47
PD4e	4287.984	---	---	4382.35	4287.46	---	---	4377.532
PD2b	4328.8	4329.4	4650.1997	4650.1999	4324.57	4324.59		

3. Description of the bifurcation diagram

A H_2 -Xe gas mixture of molar mass ratio 65.1 was selected because the parameters are well determined [12], and the separation ratio and Dufour coefficients have a similar weight [11]. The aspect ratio of the slot was fixed to $\Gamma = 8$, large enough to have a rich dynamics.

The bifurcation diagram of Fig. 1(a) is described following all the branches from left to right. Thick and thin solid lines indicate stable and unstable solutions, respectively. The primary convective state of Eqs. (8)–(10) with boundary conditions (11)–(13) is steady [11]. It becomes unstable at a Hopf bifurcation that gives rise to the main branch of POs shown in Fig. 1(a), from which a confined FC of POs starts. The steady flow is center-symmetric, i.e., it is invariant by S like with impermeable lateral sides [9]. It fulfills

$$\psi(t, 1 - x, \Gamma - y) = \psi(t, x, y), \quad (27)$$

$$\Theta(t, 1 - x, \Gamma - y) = -\Theta(t, x, y), \quad (28)$$

$$\Sigma(t, 1 - x, \Gamma - y) = -\Sigma(t, x, y). \quad (29)$$

All the bifurcated branches of POs shown below maintain the same symmetry at any time instant.

The period-doubling cascade starts at $Ra_0^T = 4030.7$. The branch of POs of double period, PD, (blue in Fig. 1) experiences a double fold and continues to larger Ra^T (already described in Ref. [11]). It loses stability at $Ra_1^T = 4155.1$ for the first time in a period doubling bifurcation (PDB) [see Fig. 1(a), (b) and (c)]. The unstable flow undergoes a second PDB at $Ra_2^T = 4533.1$, where it regains the stability, before the first saddle-node bifurcation (SNB) found at $Ra_3^T = 4539.0$ [see Fig. 1(a), (c) and (d)]. At this latter point the flow becomes unstable again when a real positive Floquet multiplier crosses the unit circle. There is a new short stability region starting in the second SNB at $Ra_4^T = 4327.5$, because the same multiplier crosses back the unit circle. A real negative multiplier crosses the unit circle at $Ra_5^T = 4328.8$, giving rise to a third PDB, and continues increasing its absolute value by increasing the continuation parameter.

The branch of quadruple period PD2a [green in Fig. 1(a)–(e)] starts at the first PDB of PD. It develops a new double fold inside the first fold of PD, and it finishes in the second PDB of PD before the SNB [see Fig. 1(d)]. The sequence of bifurcations on this PD2a branch is the same as in branch PD, i.e., it loses stability in a first PDB, and stabilizes near and before its first fold in a second PDB. Next, it loses stability in a first SNB, and soon stabilizes in the second SNB. Then, it becomes unstable in a third PDB, and finally PD2a stabilizes in a fourth PDB short before its end, finishing as stable in the PD branch [see Fig. 1(d)]. The successive critical thermal Rayleigh numbers Ra_i^T , where the branches start (Ra_0^T), double the period (Ra_i^T , $i = 1, 2, 5, 6$), undergo SNBs (Ra_i^T , $i = 3, 4$) and finish (Ra_7^T), are listed in Table 3 for most of the branches included in Fig. 1(a).

The branch of octuple period PD3a [purple in Fig. 1(a)–(c)] follows the same sequence, starting when PD2a loses stability for the first time, undergoing the double fold and ending in the PDB already described before the first SNB of PD2a. However in this case there is a second branch of octuple period PD3c, also purple in Fig. 1(a)–(e), going from the PDB after the second fold of PD2a to the fourth PDB of PD2a before its end in PD. It undergoes a new double fold between these endpoints. The bifurcations on PD3c reproduce those of the other commented branches. The same behavior could be described for the branches of sexdecuple period PD4a, PD4e [orange in Fig. 1(a)–(c)], the first starting and finishing in the PDBs near the beginning and the end of PD3a, respectively, and the other starting and finishing in the PDBs near the two folds of PD3a, after completing two new folds each of them. The number of branches of period multiple of the initial one doubles at each pair of PDBs as a consequence of this dynamics. Moreover, the intervals of stability are shorter at each PDB, as expected in a FC, and increasingly smaller increments of the parameter for larger periods are needed to determine the bifurcations (see Table 3). For this reason, and also because the Floquet multiplier of larger modulus becomes more unstable at each PD, only the four pieces of the PD4 branches of sexdecuple period, starting on branch PD3c, have been computed. One of them is labeled PD4c in Fig. 1(c) and (e), and another (or perhaps the same) tagged PD4 in Fig. 1(d). However, it has been checked that each PD4 piece soon loses stability in new PDBs as the others; PD4c at $Ra^T = 4294.825$, and PD4 at 4528.994 . Those starting near the left and right saddle-nodes of PD3c become unstable at $Ra^T = 4299.8606$ and $Ra^T = 4391.8887$, respectively. Therefore, all the computations indicate that the new bifurcations would give rise to eight new branches of period $32T$, and so on.

A second branch of POs of quadruple period, labeled PD2b in Fig. 1(a), arises in the PDB, after the second fold of PD [see Table 3, and Fig. 1(a) and (f)]. It soon loses stability in another PDB, and it experiences a double fold, as the preceding branches. From PD2b

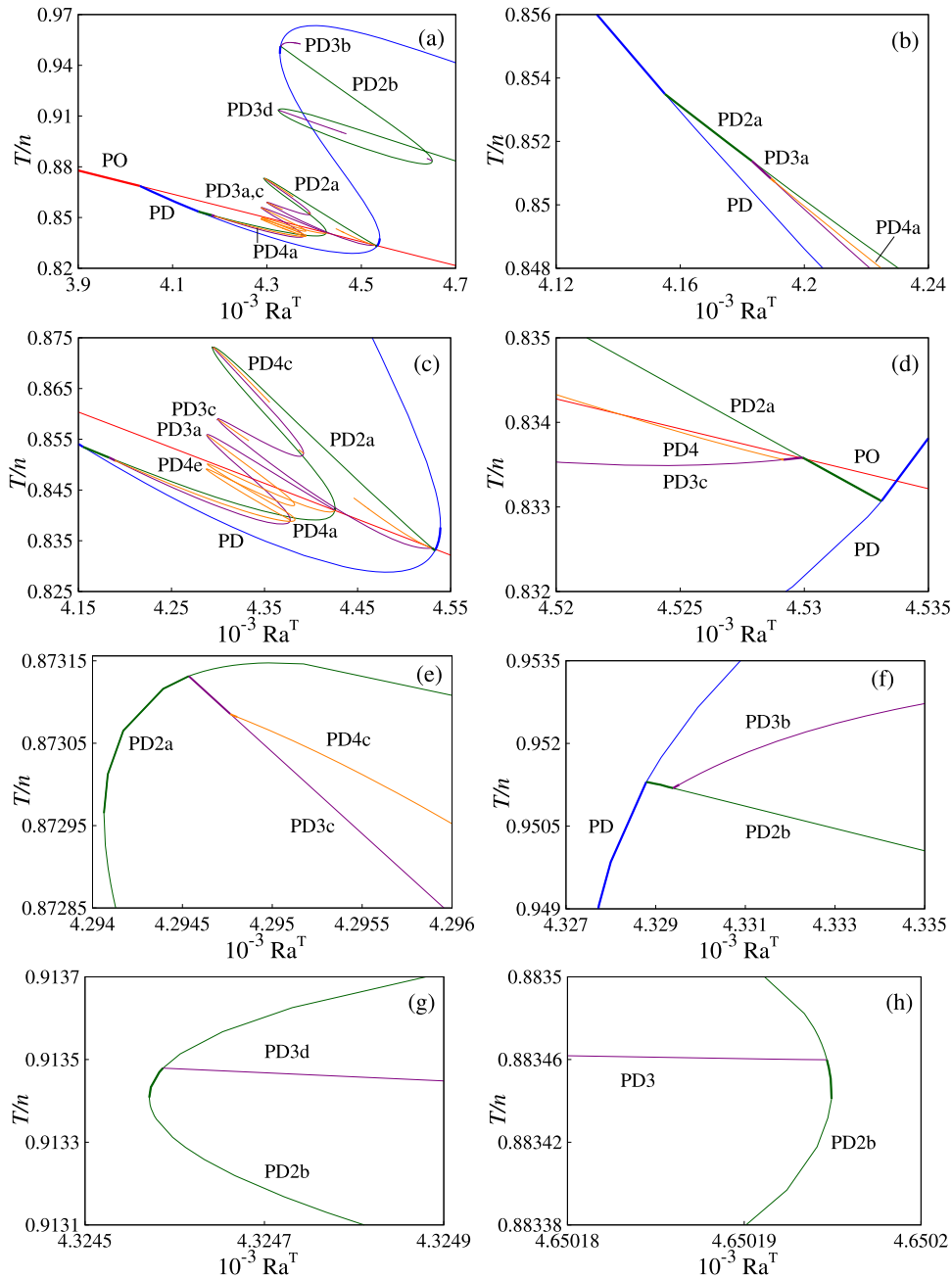


Fig. 1. (a) Period T/n , with $n = 1, 2, 4, 8, 16$, of the main branch (PO) of periodic orbits, and branches of double (PD), quadruple (PD2), octuple (PD3), and sexdecuple (PD4) period versus Ra^T in red, blue, green, purple and orange, respectively. Thick and thin solid lines indicate stable and unstable solutions, respectively. From (b) to (h) details of (a) showing some PDBs on these branches.

two new branches of octuple period, labeled PD3b and PD3d, arise at PDBs that exist on branch PD2b [see Fig. 1(f) and (g)]. One of them would finish on the PDB shown in Fig. 1(h), and the other would continue to higher Ra^T . The main difference between the dynamics of the nested FCs inside the lower fold of the PD branch and the upper one is that one of the branches of any period multiple of the initial would continue to higher Ra^T after a double fold as PD and PD2b do, but the behavior is basically the same.

A sequence of direct and inverse period doubling cascades by increasing Ra^T was described in Ref. [22], however the doubling of the number of branches within each fold, after each pair of PDBs and SNBs, was not found.

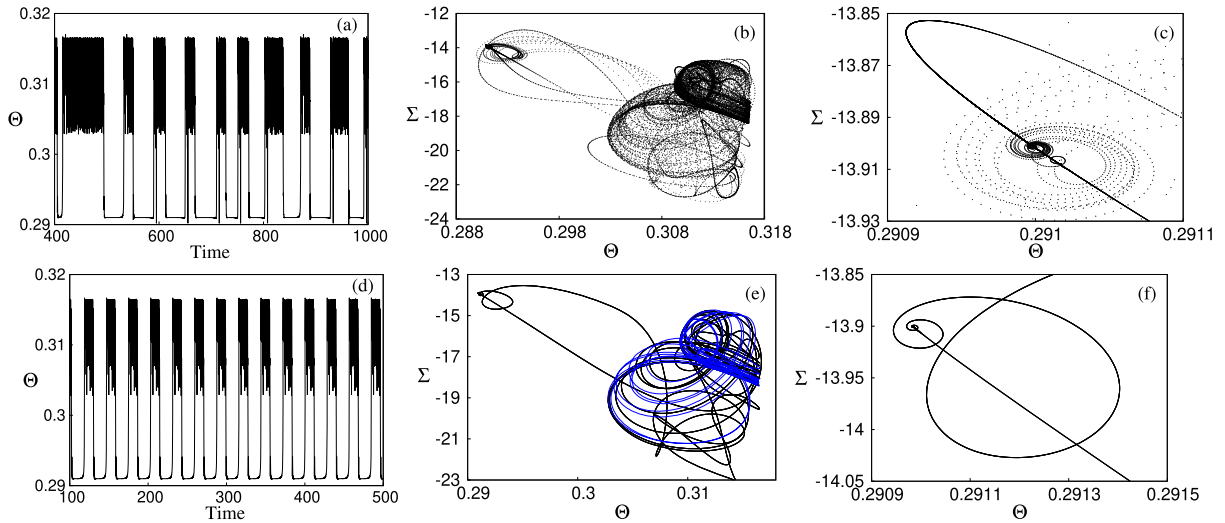


Fig. 2. Nearly-heteroclinic orbits: (a)-(c) started with IC taken on branch PD before the first fold at $Ra^T = 4.351964115 \times 10^3$. (d)-(f) started with IC taken on branch PD3d at $Ra^T = 4.350094707 \times 10^3$. Figure (e) also includes the periodic IC of octuple period in blue color. The transients are not included.

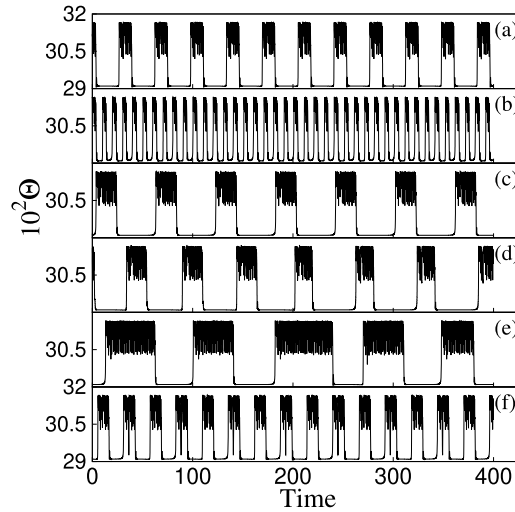


Fig. 3. Temporal evolution of some NHOs with ICs taken on different branches of Fig. 1 near $Ra^T = 4.35 \times 10^3$. Branch (a) PD, upper part at 4.3528672×10^3 , (b) PD2a, middle part at 4.3494489×10^3 , (c) PD3a, upper part at 4.350731753×10^3 , (d) PD3c lower part at 4.349963190×10^3 , (e) PD3b at 4.350074605×10^3 , and (f) PD3c, upper part at 4.351434115×10^3 . The transients have been removed.

4. Dynamics in the interval of Ra^T where the periodic orbits are unstable

Direct numerical simulations were performed to complete the study of the dynamics in the interval of parameters where the POs of Fig. 1(a) are unstable. Some points on the branches of POs, near $Ra^T = 4.35 \times 10^3$, were taken as initial conditions (IC).

Fig. 2 shows examples of the temporal evolutions and two-dimensional phase map projections of a chaotic nearly-heteroclinic orbit (NHO), found with an IC taken on the branch PD, before the first SNB, and a periodic NHO found with an IC taken on branch PD3d.

The temporal sequences display Θ at $(x, y) = (0.48, 7.78)$ approximately, and the associated phase maps display Θ at $(0.48, 7.78)$ versus Σ at $(0, 48, 7.20)$. The intervals where the solutions of Fig. 2(a) and (d) are quiescent (almost time independent) correspond to the left part of Fig. 2(b) and (e). They illustrate the approach of the trajectories to fixed points. It is shown in detail in Fig. 2(c) and (f). The intervals of strong time dependence belong to the right part of the phase maps of Fig. 2(b) and (e). The latter also includes the projection of the unstable octuple periodic orbit from which solution (d)-(f) was obtained. It shows the approach of the NHOs to the POs of the FCs. All of these POs of doubled, quadrupled, etc. periods would superpose to that represented in Fig. 2(e) if they were included, but with a different number of loops.

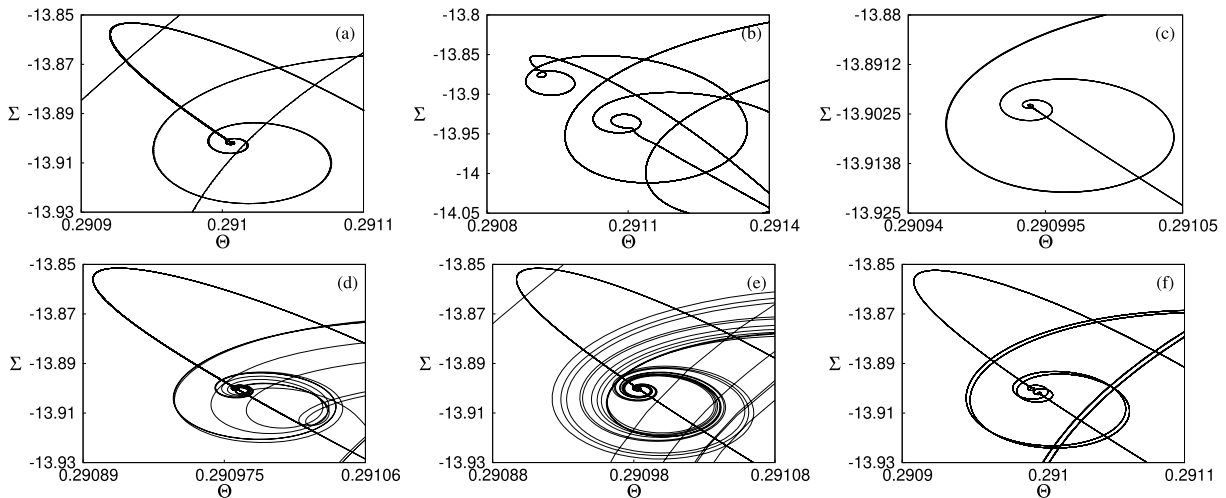


Fig. 4. Details of the phase maps of the NHOs of Fig. 3 near the saddle-foci.

Several ICs taken on other branches of Fig. 1 for Ra^T near 4.35 lead to the type of NHOs shown in Figs. 3 and 4. Three types of trajectories can be recognized from the time evolution; single loop limit cycles (which include a quiescent stage followed by another oscillatory) [Fig. 4(a) and (c)], double loop limit cycles [Fig. 4(b) and (f)], and chaotic orbits [Fig. 4(d) and (e)]. These phase-maps reveal that the three types of orbits approach a saddle-focus, although at first glance it might seem that two fixed points can form part of the NHOs in Fig. 4(b) and (f) (see below).

The contour plots of the NHOs during the two stages of a cycle show that the structure of the flow is completely different during the time independent and dependent intervals. Consequently, to understand their origin, the curve of unstable fixed points (SS3 from now on), which become evident in the NHOs, was computed by using the continuation method. It was started from a point of the NHO obtained with IC the point of the lower part of branch PD at $Ra^T = 4.3519641 \times 10^3$. For this purpose, the direct numerical simulation was stopped during a quiescent stage. Fig. 5, shows the diagram of the Nusselt number versus Ra^T of some equilibria and POs that contribute to the existence of the NHOs, and others found in the range of parameters where SS3 exists. Branch SS3 is an elongated isola that extends along the range where the FCs were found. The two folds of SS3 are located at $Ra^T = 2.72890096 \times 10^3$ and 9.39567027×10^3 , and two Hopf bifurcations were found in the upper branch past $Ra^T = 8.872 \times 10^3$, before the right fold. All solutions on SS3 are unstable. Notice that in this representation the period-doubling cascades fall inside the double fold of the PD branch [which is better seen in Fig. 5(b)]. They are not included in the diagram because they are indistinguishable in sketches of mean-values extended over large ranges of the parameter. The SS2 branch (black) is the main branch of fixed points, from which the main curve of periodic orbits, PO (red), bifurcates [11]. Several orbits were calculated by taking saddle-foci of SS3 as IC. The trajectories follow the unstable manifold, and quickly tend to NHOs like those of Fig. 4, without any other transient.

In addition Fig. 5 includes in grey color the branch PO2 bifurcated from PO in a pitchfork bifurcation of periodic orbits (PPO) (already described in [11]), and two branches of periodic orbits detected in the same region of the parameter space, which are stable in short intervals of Ra^T . Branch PO3 keeps the primary center-symmetry of the flow, it has a period that decreases by increasing the parameter, and it is stable between $Ra^T = 9.4766 \times 10^3$ and 1.10754×10^4 , having a period about 0.5. It loses and gains stability at two PPOs. By decreasing the parameter it undergoes a homoclinic bifurcation near $Ra^T = 5.7233 \times 10^3$. Branch PO4 is stable between $Ra^T = 7.4469728 \times 10^3$ and 7.611624×10^3 , it does not preserve the symmetry, and its period also decreases when the parameter increases, but it octuples that of PO3 in its stability interval. The branch loses stability for the first time in a PDB. It has three other small stability intervals near $Ra^T = 7.96 \times 10^3$, and around two SNs placed at $Ra^T = 8.49 \times 10^3$ and 8.58×10^3 . These branches were computed from stable IC obtained by direct simulation, and the origin of PO4 was not investigated.

The contour plots of ψ , Θ , and Σ , during the intervals in which the NHO are almost time independent, are shown on the left side of Fig. 6. They are compared to the same contour plots of a fixed point of SS3 at almost the same Ra^T on the right side. From the structure and symmetry of both solutions, it seems clear that these intervals are due to the approach of the trajectory to a fixed point of SS3. However, it is impossible to assure at first sight if the trajectory approaches the upper or the lower part of the isola because the contour plots look very similar.

Fig. 7 shows several contours plots of the NHO calculated with IC taken on branch PD2a at $Ra^T = 4.3494489 \times 10^3$, during the oscillations. Their comparison with those of a period of the unstable POs in the region of PDs shows the same structure (see Figs. 2–4 of Ref. [11]), with two strong vortices oscillating vertically from the upper and lower sides to the center of the cavity. Only when the trajectory starts to approach the saddle-focus the contour plots differ significantly from those of Fig. 7.

Some additional tests were done since the similarity of the contours plots is not conclusive to determine the fixed points involved. Fig. 8(a) shows the euclidean norm of the difference of the state variables at the collocation points of a NHO and two points of SS3, both at $Ra^T = 4.3494489 \times 10^3$.

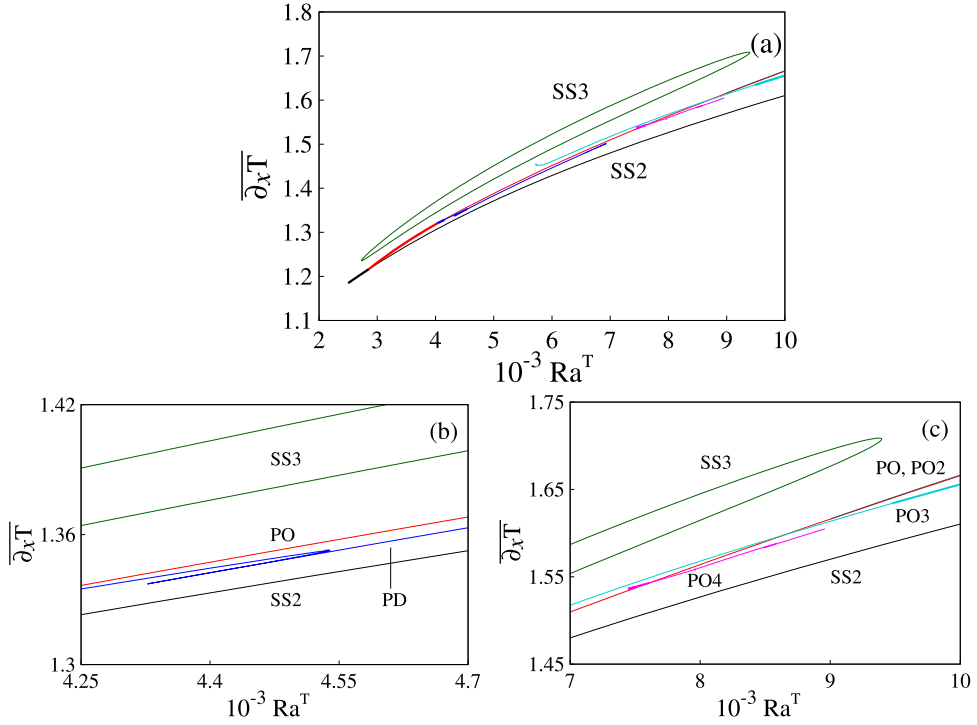


Fig. 5. (a) Diagram of the SS2 branch (black) and SS3 isola (green) of steady states, and of the PO (red) and PD (blue) periodic orbits. (b) detail of (a) in the zone where the FCs and NHOs are located. (c) detail of (a) out of the range of the FCs. PO2 (grey), PO3 (cyan), and PO4 (magenta) Thick and thin lines are stable and unstable solutions respectively.

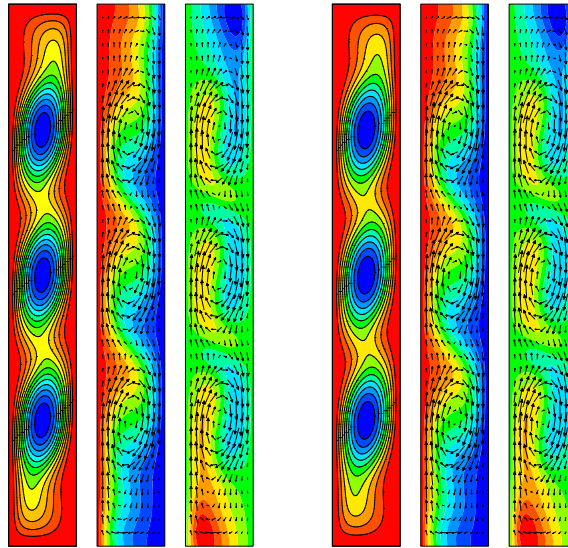


Fig. 6. From left to right contour plots of ψ , Θ and Σ of a NHO near a saddle-focus at $Ra^T = 4.35 \times 10^3$, and of the fixed point of the upper part of SS3 at $Ra^T = 4.35009 \times 10^3$.

The black and blue lines correspond to the equilibria on the upper and lower parts of the isola, respectively. The latter never approaches zero indicating that the NHO only comes close to the fixed point located on the upper section. This fact is in agreement with the leading (closer to the imaginary axis) spectra of the equilibria of SS3. It has been checked that near $Ra^T = 4.35 \times 10^3$, for which the phase-maps are plotted, the spectra have a positive real eigenvalue and a complex pair of stable eigenvalues in the upper part of the isola, namely the equilibria are of saddle-focus type. In contrast, they have two unstable real eigenvalues and a stable real one, followed by several stable complex pairs, in the lower part, i.e., the equilibria are saddle points with a two-dimensional unstable

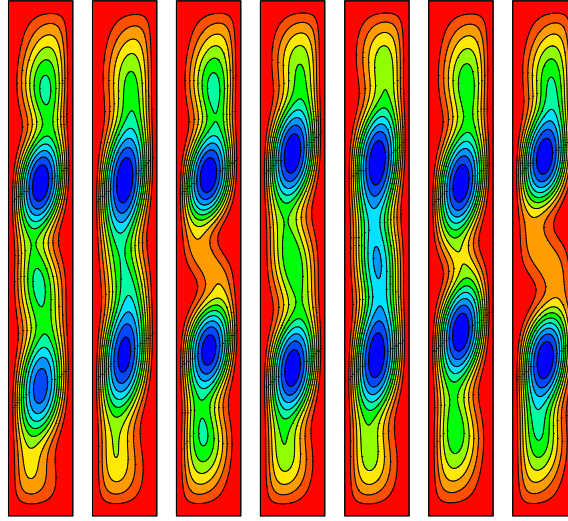


Fig. 7. Some snapshots of ψ of a NHO with IC taken on PD2a at $Ra^T = 4.3494489 \times 10^3$ during the oscillations.

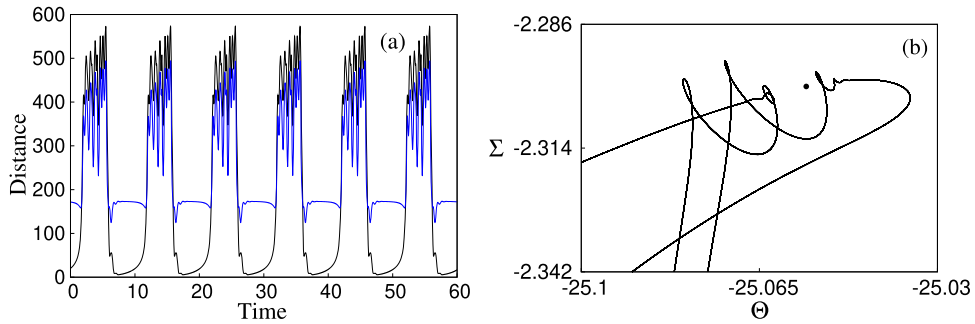


Fig. 8. (a) Distance from the NHO, computed with IC taken from branch PD2a at $Ra^T = 4.3494489 \times 10^3$, between the two folds, to the fixed points of SS3 at the same Ra^T . The black and blue lines give the distance to the equilibria on the upper and lower parts, respectively. (b) Detail of the phase map of the same NHO including the saddle-focus of SS3 at the same value (black dot).

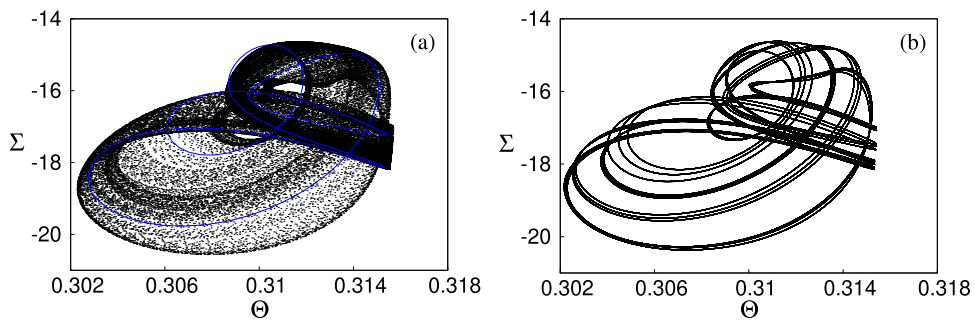


Fig. 9. (a) Phase map of an unstable PO of branch PD at $Ra^T = 4.273893533 \times 10^3$ (blue), superposed to that of the chaotic attractor found at $Ra^T = 4.288443475 \times 10^3$ from an IC on branch PD4e. (b) Fifteen-loop stable periodic orbit found with IC the PO of double period shown in (a) at the same Ra^T .

manifold. The saddle-foci in the upper part extent from $Ra^T \approx 3.76 \times 10^3$ to 8.873×10^3 , covering the interval of Ra^T of the folds of PD and PD2a and beyond.

Fig. 8(b) shows a different projection than that of Fig. 4(b), including the position of the saddle-focus. The location of the saddle point falls out of this plot. This view clarifies the meaning of Fig. 4(b) and (f). These orbits approach a saddle-focus from the two sides of its stable manifold, and move away from the vicinity of the equilibrium following its one-dimensional unstable manifold in the same direction every two loops. The only difference between the maps of Fig. 4(b) and (f) is the distance at which the NHOs

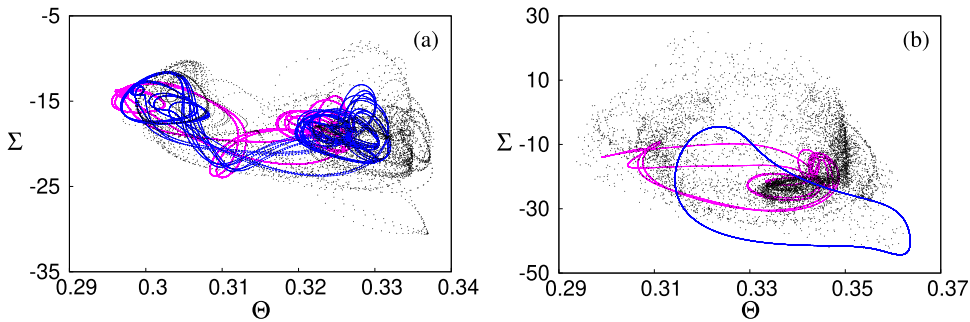


Fig. 10. Phase maps beyond the interval of FCs. (a) NHO at $Ra^T = 5.1 \times 10^3$ (magenta), PO of sextuple period at 5.5×10^3 (blue), and chaotic orbit at 6.0×10^3 (black). (b) PO of double period of PO4 at 7.5×10^3 (magenta), chaotic attractor at 9.0×10^3 (black), and periodic orbit of PO3 at 1.0×10^4 (blue).

approach the equilibria. Fig. 4(a) and (c) show two orbits that approach the saddle-focus always from the same side, in contrast to the double-looped or the chaotic NHOs.

Other ICs away from $Ra^T = 4.35 \times 10^3$ give rise to different dynamics. Fig. 9(a) shows, in blue color, a phase map projection of the unstable orbit of the PD branch at $Ra^T = 4.273893533 \times 10^3$, superposed to the chaotic attractor obtained with an IC taken on the PD4e curve at $Ra^T = 4.288443475 \times 10^3$. It is close the fold of branch PD3a where it arises, and when PD4e has already become unstable. In addition, Fig. 9(b) shows a stable orbit of fifteen loops found at $Ra^T = 4.273893533 \times 10^3$ taking as IC the PO of simple period shown in Fig. 9(a). Notice that the three solutions are close to the limit where the left saddle-nodes of the branches of periodic orbits tend to align. The influence of the saddle-foci of SS3 disappears in this zone. The comparison of the contour plots and frequencies (from the Fourier spectra) indicates that this dynamics can be simply linked to the cloud of unstable POs associated to the FCs.

In addition, other random ICs were employed with Ra^T outside the interval of the FCs and beyond the SS3 isola. Periodic orbits with broken center-symmetry and chaotic attractors were found for $Ra^T > 5.1 \times 10^3$ (see Fig. 10(a)). However, this chaos is still affected by the existence of SS3, unlike the previous case. By increasing the parameter the influence of the saddle-foci of SS3 gradually weakens and finally the Shil'nikov-like dynamics related to SS3 disappears. Some intervals of new stable periodic orbits and also chaotic attractors related to the new orbits were obtained for $Ra^T > 6.4 \times 10^3$ [see Figs. 10(b) and 5(b)].

5. Summary and conclusions

The time dependent double-diffusive convection of a H_2 -Xe gas mixture has been analyzed. It provides a real-life example where both FCs and Shil'nikov-like dynamics can be observed. The main feature is the loss of stability of the main branch of POs in a PDB [11], followed by a series of nested FCs, which give rise to an increase of the number of branches of POs following a geometric progression. This is so because each branch has two folds, and the successive bifurcations along each of them are: starting point, PDB1 (stability loss), PDB2 (stability gain), SNB1 (stability loss), SNB2 (stability gain), PDB3 (stability loss), PDB4 (stability gain), ending point. During the long instability intervals between PDBs and SNBs, or vice versa, the real negative or positive multipliers of larger absolute value quickly increase and decrease their modulus and reach values as large as 10^5 in the PD4 branches. The change becomes faster at each period doubling, and therefore the stability zone is shortened. Both factors, together with increasingly longer periods, make the computation so slow that it would be necessary to apply a multiple shooting parallel method with a suitable preconditioner [23] to accelerate the continuation of the periodic orbits. Even so, it would be impossible to carry out the calculations much further. The PDBs and the closest SNBs would probably collapse giving rise to codimension-two bifurcations by slightly changing some of the parameters.

Some direct numerical simulations have been done to determine the stable dynamics coexisting with the unstable periodic orbits. The first kind of solutions found are orbits (periodic or chaotic) nearly heteroclinic to a hyperbolic equilibrium on the isola SS3 and to one or several (heteroclinic chain) unstable POs of the FCs. It is clear from Fig. 2 and from the comparison of the contour plots that the trajectories connect the unstable manifold of a PO with the stable manifold of an equilibrium of SS3 and vice versa. These equilibria are not related to the origin of the FCs. The spatial patterns of the NHOs close to SS3 and to the POs are completely different (see Figs. 6 and 7). The existence of the orbits described agrees with the theoretical study by J.D.M. Rademacher [24], who analyzed homoclinic orbits near codimension-1 and -2 heteroclinic cycles between an equilibrium and a periodic orbit for ordinary differential equations in three or higher dimensions. In addition, this numerical study is a new application compatible with the theories developed in Refs. [25,26], in which codimension-two mechanisms for the existence of equilibrium-to-periodic heteroclinic bifurcations were studied. By integrating the FitzHugh-Nagumo equation in the former, and in the latter by means of the construction and analysis of geometric return maps to partially unfold the dynamics occurring near a heteroclinic cycle between a hyperbolic equilibrium and a hyperbolic periodic orbit under the conditions of the leading spectra found here.

The second kind of chaotic solutions found involving the POs is consistent with the appearance of chaos due to FCs, since windows of POs of periods different to those of the sets of period doublings, but with the same spatio-temporal behavior, were found. However, the mechanism of this complex dynamics would require a theoretical study.

The saddle quantity, σ , of a saddle-focus is defined in terms of the leading eigenvalues λ_i , with $i = 1, 2, 3$, as $\sigma = \lambda_1 + \Re(\lambda_{2,3})$. On the one hand, $\sigma < 0$, with $\lambda_2 = \bar{\lambda}_3$, for the saddle-focus on the upper part of SS3 in the range of existence of the FCs, and the leading eigenvalues are real in its lower part with $\sigma = \lambda_2 + \lambda_3 > 0$ in the same interval. Shil'nikov's theorems [27–29] predict in both cases the existence of a unique limit cycle in a neighborhood of a homoclinic orbit, irrespective of the sign of σ in the second case. Our calculations are in agreement with the theory in the sense that when a single limit cycle (periodic NHOs) is found, for a fixed Ra^T , it is independent of the IC. Then the chaotic behavior found in some of the simulations represented in Figs. 2 and 3 cannot be associated to Shil'nikov's chaos. On the other hand, there are many sets of unstable POs, coming from the FCs in the interval of Ra^T of these temporal evolutions. A trajectory starting close to a saddle-focus will follow close to its one-dimensional unstable manifold and can approximate one of these chaotic regions of POs, coming close to one or several of them, and escaping from the region along the unstable manifold of one of the POs. The orbit can turn back to the equilibrium along its codimension-one stable manifold, starting a new loop. This would cause the chaotic behavior observed in some cases.

It seems that the periodic dynamics found here could be reproduced with a low-dimensional model, despite we are dealing with a high-dimensional discretization of a PDE system. Moreover, the local dynamics described should be quite generic because it has been obtained by moving a single parameter, the unstable orbits have a single Floquet multiplier outside the unit circle, and the center symmetry of the system plays no role in the sequence of FCs and SNs. However, the global dynamics (NHOs) described is more complex because the sequence of nested FCs gives rise to an strong dependence on Ra^T . For these reasons, additional theory, return maps, or reduced models, out of the scope of this paper, are needed to help to understand the final steps of the nested FCs, and the mechanism of connection of the saddle-foci with the cloud of POs.

The results found seem to indicate that in a perfect scenario the FCs and NHOs are constrained to the range of parameters delimited by the SNs of the branches of POs, which tend to align [see Fig. 1(a)]. Each fold of the PD branch would bound a nest of FCs. In the present case the finite aspect ratio of the box limits the number of these folds.

CRediT authorship contribution statement

Juan Sánchez Umbria: Writing – original draft, Visualization, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization; **Marta Net:** Writing – original draft, Visualization, Software, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Data availability

Data will be made available on request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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