Azimuthal waves and their stability in externally heated rotating spherical shells

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Summary. Azimuthal waves appearing in the thermal convection of a pure fluid contained in a spherical shell with both boundaries at different temperatures are studied. They are computed by using continuation methods as steady solutions in the reference system of the wave. There stability is also studied, and the secondary bifurcations to modulated waves are detected.

Equations for the waves in the spherical shells

Consider a spherical shell of inner and outer radii r_i and r_o, rotating at an angular velocity $\mathbf{\Omega}$, filled with a homogeneous fluid of density ρ, thermal diffusivity κ, thermal expansion coefficient α, kinematic viscosity ν, and in the presence of a radial gravitational field $\mathbf{g} = -\gamma\mathbf{r}$. In the Boussinesq approximation κ, α and ν are considered constants, and $\rho = \rho_0(1-\alpha(T-T_0))$ is assumed to vary linearly with the temperature T just in the gravity term. Therefore the velocity field $\mathbf{v}$ satisfies Navier-Stokes equations and it is divergence-free. We assume constant temperature at the boundaries, ΔT being the difference of temperatures between the two spheres.
The non-dimensional parameters appearing in the equations are the radius ratio η, and the Ekman, E, Rayleigh, R, and Prandtl, σ, numbers defined as

$$\eta = r_i/r_o, \quad E = \nu/\Omega d^2, \quad R = \gamma\alpha\Delta T d^4/\kappa\nu, \quad \sigma = \nu/\kappa \quad \text{with} \quad \Omega = |\mathbf{\Omega}|, \quad \text{and} \quad d = r_o - r_i. \tag{1}$$

The Ekman number is related to the Taylor number by $E = Ta^{-1/2}$. The conduction state given by $\mathbf{v} = 0$ and $T_c(r) = T_0 + R\eta/\sigma(1-\eta)^2 r$, where T and r are now non-dimensional variables, is a solution for any value of the parameters. When it becomes unstable the bifurcation is usually a Hopf bifurcation giving rise to azimuthal waves.
The equations governing the dynamics of the fluid are written in spherical coordinates (r,θ,φ) (θ being the colatitude, and φ the longitude), in the rotating frame of reference of the spheres, and in terms of two scalar potentials (toroidal and poloidal) for the velocity, i.e, $\mathbf{v} = \nabla\times(\Psi\mathbf{r}) + \nabla\times\nabla\times(\Phi\mathbf{r})$. The equations for the potentials are obtained by applying the operators $\mathbf{r}\cdot\nabla\times$ and $\mathbf{r}\cdot\nabla\times\nabla\times$ to Navier-Stokes equations. The potentials and the perturbation of the temperature from the conduction state are expanded in spherical harmonic series up to degree L as

$$(\Psi,\Phi,\Theta)(t,r,\theta,\varphi) = \sum_{l=0}^{L}\sum_{m=-l}^{l}(\Psi_l^m,\Phi_l^m,\Theta_l^m)(t,r)Y_l^m(\theta,\varphi),$$

with $\Psi_l^{-m} = \overline{\Psi_l^m}$, $\Phi_l^{-m} = \overline{\Phi_l^m}$, and $\Theta_l^{-m} = \overline{\Theta_l^m}$. Moreover, to have the two potentials completely determined we can choose $\Psi_0^0 = \Phi_0^0 = 0$.
If the operator D_l is defined as $\mathcal{D}_l = \partial_{rr}^2 + (2/r)\partial_r - l(l+1)/r^2$, the equations for the amplitudes are

$$\partial_t\Psi_l^m = \mathcal{D}_l\Psi_l^m + \frac{1}{l(l+1)}\left[2E^{-1}\left(im\Psi_l^m - [Q\Phi]_l^m\right) - [\mathbf{r}\cdot\nabla\times(\boldsymbol{\omega}\times\mathbf{v})]_l^m\right], \tag{2}$$

$$\partial_t\mathcal{D}_l\Phi_l^m = \mathcal{D}_l^2\Phi_l^m - \Theta_l^m + \frac{1}{l(l+1)}\left[2E^{-1}\left(im\mathcal{D}_l\Phi_l^m + [Q\Psi]_l^m\right) + [\mathbf{r}\cdot\nabla\times\nabla\times(\boldsymbol{\omega}\times\mathbf{v})]_l^m\right], \tag{3}$$

$$\partial_t\Theta_l^m = \sigma^{-1}\mathcal{D}_l\Theta_l^m + \sigma^{-1}l(l+1)R\eta(1-\eta)^{-2}r^{-3}\Phi_l^m - [\mathbf{v}\cdot\nabla\Theta]_l^m. \tag{4}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{v}$ is the vorticity, and the operator Q is defined by its action on a function f expanded in spherical harmonics as

$$[Qf]_l^m = -l(l+2)c_{l+1}^m D_{l+2}^+ f_{l+1}^m - (l-1)(l+1)c_l^m D_{1-l}^+ f_{l-1}^m, \quad \text{with} \quad c_l^m = ((l^2-m^2)/(4l^2-1))^{1/2}, \tag{5}$$

and $D_l^+ f = \partial_r f + lf/r$. On the boundaries $r_i = \eta/(1-\eta)$ and $r_o = 1/(1-\eta)$, stress-free ($\Phi_l^m = \partial_{rr}^2\Phi_l^m = \partial_r(\Psi_l^m/r) = 0$) or non-slip ($\Phi_l^m = \partial_r\Phi_l^m = \Psi_l^m = 0$) boundary conditions may be selected for the velocity field. Perfectly conducting ($\Theta_l^m = 0$) boundaries are used for the temperature (see [1, 2, 3] for more details on the formulation).
The above system, will be written as

$$L_0\partial_t u = Lu + B(u,u), \tag{6}$$

where $u = u(t,r,\theta,\varphi)$ is a vector containing the values of the amplitudes at a mesh of collocation points in the radius, and L and B are, respectively, linear and bilinear operators, with L depending on all the parameters of the problem. As the dependence of the solution will be studied by fixing all of them except R we will make explicit the dependence on a single parameter ($\lambda = R$ in our calculations). Suppose that at $\lambda = \lambda_c$, the solution $u = 0$ becomes unstable, and a branch of azimuthal waves starts there. Then, at this value of λ, there are a vector v_c and a constant ω_c such that $i\omega_c L_0 v_c = Lv_c$.
The waves $u(t,r,\theta,\varphi) = \tilde{u}(r,\theta,\tilde{\varphi})$, solutions of (6), with $\tilde{\varphi} = \varphi - \omega t$, verify the equation $\omega L_0\partial_{\tilde{\varphi}}\tilde{u} + L\tilde{u} + B(\tilde{u},\tilde{u}) = 0$, or, by deleting the tildes,

$$F(u,\omega,\lambda) = \omega L_0\partial_\varphi u + L(\lambda)u + B(u,u) = 0. \tag{7}$$

This equation must be supplemented by adding a phase condition. We use the condition $G(u) =< u, \partial_\varphi u_c >= 0$, where $< \cdot, \cdot >$ is the $\mathcal{L}^2$ product, and u_c is a reference solution (the eigenvector, $u_c = v_c$, at $\lambda = \lambda_c$, or a previously computed solution). It is a necessary condition for $||u - u_c||_2^2$ to be minimal with respect to the phase.
The action of the Jacobian $(\partial_u F, \partial_\omega F, \partial_\lambda F)(u, \omega, \lambda)$ on (v, ζ, μ) is

$$\partial_u F(u, \omega, \lambda)v + \partial_\omega F(u, \omega, \lambda)\zeta + \partial_\lambda F(u, \omega, \lambda)\mu = \omega L_0 \partial_\varphi v + \zeta L_0 \partial_\varphi u + L(\lambda)v + \mu L^{(2)} u + B(u, v) + B(v, u),$$

due to the dependence of L on λ, which has the form $L(\lambda) = L^{(1)} + \lambda L^{(2)}$. The action of the Jacobian $\partial_u G(u)$ on (v, ζ, μ) is $\partial_u G(u)v =< v, \partial_\varphi u_c >$. During the continuation process to study the dependence of the waves with λ, linear systems of equations with matrices

$$\begin{pmatrix} \partial_u F & \partial_\omega F & \partial_\lambda F \\ \partial_u G & 0 & 0 \\ w_u^\top & w_\omega & w_\lambda \end{pmatrix} \tag{8}$$

must be solved. The last row comes from the pseudo-arclenght condition They will be preconditioned by matrices of the form

$$\begin{pmatrix} \omega_p L_0 \partial_\varphi + L_p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with $L_p = L(\lambda_p)$ and ω_p being the operator L and the frequency of the wave at a previous step. Since L is block-tridiagonal, due to the operator Q (5), it is possible to solve this latter system efficiently [4].
To study the stability of the waves there are two possibilities, finding the leading eigenvalues of the Jacobian (8), which requires some kind of transformation (Cayley or shift-invert, for instance), or integrating the linearized equations about the wave, which does not require transformations, but it is more expensive.

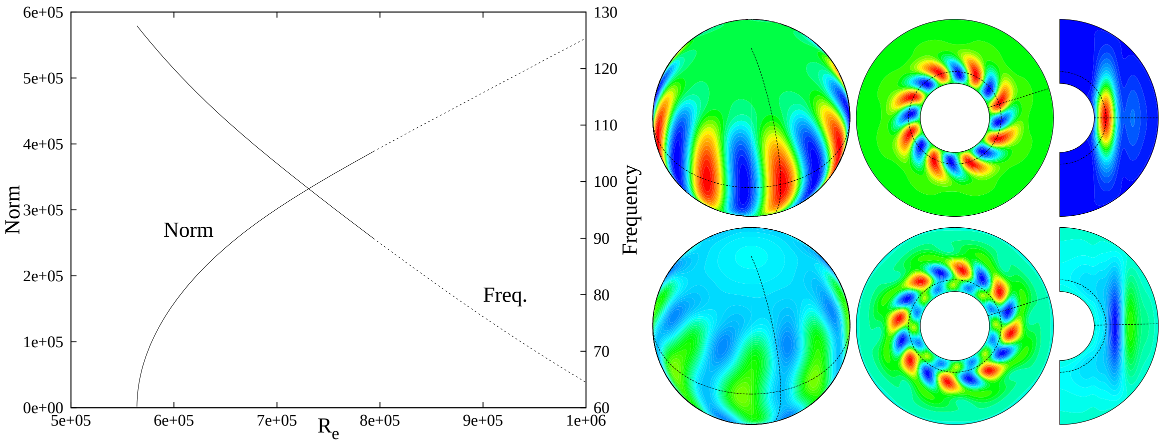

Figure 1: On the left, norm of the solution, $||u||_2$, and frequency, ω, of the wave as a function of the Rayleigh number. On the right (up), solution at the bifurcation point ($R = 7.936 \times 10^5$), and (down) imaginary part of the leading eigenfuction giving rise to a modulated wave.

Preliminary results and conclusions

Fig. 1, computed with non-slip boundary conditions, shows a typical plot of the curve of solutions and the frequency as a function of the control parameter R. The rest of the parameters have been kept constant to the values $\eta = 0.35$, $\sigma = 0.1$, and $E = 3.54 \times 10^{-5}$ ($Ta = 8 \times 10^{-8}$). The discretization dimensions are $n_r = 32$ (the number of radial collocation points) and $L = 8 \times 16 = 128$ (only the first 16 spherical harmonics of order multiple of 8 are stored). The contour plots on the right represent the level curves of the perturbation of the temperature at the three sections indicated with dashed lines. The upper plots correspond to the bifurcation point which gives rise to a quasiperiodic solution (for perturbations with the same azimuthal wavelength). The lower ones show the imaginary part of the leading eigenfunction at the bifurcation point.

References

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